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A HARMONIC L_4 ORBIT

FOR

THE VERY RESTRICTED FOUR-BODY PROBLEM

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ABSTRACT

A method of general perturbations utilizing Chebyshev series is used to investigate motion in the vicinity of the L_4 triangular point of the earth-moon system. The model used is that of the very restricted four body problem for the earth-moon-sun system. A harmonic orbit, in the numerical sense, with respect to a rotating coordinate frame centered at L_4 is found. The period of this harmonic orbit, being equal to the period of the disturbing force, is the same as the moon's synodic period. This orbit remains within 6860 km of the L_4 point. It describes two different size loops about L_4, the smaller one traversed in 36 percent of the period. The disturbing force, being nearly periodic with half the moon's synodic period, gives rise to another orbit about L_4 which is nearly periodic with half the synodic period of the moon. This orbit remains within 4574 km of the L_4 point for twelve periods investigated. Deviations from the mid orbit during this time is less than 381 km.

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1. INTRODUCTION

In some recent literature interest has been shown in the problem of the influence of the sun on motion close to the libration points of the earth-moon system as well as motion about the earth and the moon itself. One possible model for the earth-moon-sun system in which the problem might be considered has been proposed by Su-Shu Huang (1960), who called it the "very restricted four-body problem." Here the earth and moon describe circular orbits relative to one another, and their center of mass describes a circular orbit around the sun; all these orbits are Keplerian, lie in a plane, and no perturbations are considered. Using this model Huang studied the motion of a fourth body of an infinitesimal mass in a similar manner as in the restricted three-body problem. He concludes this model gives a general idea of where the fourth body could or could not go under given initial conditions when they are no longer very near the earth. Using a similar model Cronin et al. (1964) proved that under certain conditions the fourth body has a periodic motion, relative to a rotating coordinate frame, near each of the libration points of the restricted three-body problem. Their proof is based upon assumptions concerning the masses and distances of the bodies which are not satisfied by the earth-moon-sun system.

Siferd (1965) used Huang's model for the earth-moon-sun system to generate some periodic orbits. Using a numerical integration procedure, the equations of motion for the very restricted four-body problem were iterated upon utilizing a digital computer until some periodic orbits were obtained. By this technique eight periodic orbits, in the numerical sense, with a respect to a rotating coordinate system were found. Three periodic orbits were around the earth, three were around the moon, and two were around the earth-moon libration point (L_1). No periodic orbits near the triangular points were obtained.

Danby (1965) investigated the influence of the sun on motion close to the triangular points of the earth-moon system. He felt the very restricted four-body model inadequate for his investigation and therefore used a model in which the secular perturbations of the moon due to the sun were retained. The results may be said to strengthen the hope that stable motion around the triangular points of the earth-moon system is possible. Other investigators include Tapley, et al. (1963 and 1965) who used a model similar to the very restricted four-body model except the moon's orbit is inclined with respect to the earth-sun plane. The equations of motion for a particle near the triangular points of the earth-moon system are numerically integrated on a digital computer for various initial conditions. One result indicates that a particle placed initially at a triangular point (L_4) with zero relative velocity has an envelope of motion, centered at L_4, going through a mode of expansion to a value of one earth-moon distance for the major axis followed by a mode of contraction to a value of 1/8 earth-moon distance for the major axis. The envelope repeats this sequence several times during the 2500-day period investigated. The nature of these data suggests that such a motion may persist for a period of time much longer than that considered in the study.

The present paper uses the very restricted four-body problem model as proposed by Huang for the earth-moon-sun system. The merits of this model for the earth-moon-sun system are still in doubt; however, it has been used in this study as a first attempt to find orbits which remain near triangular points of the earth-moon system. Using a technique proposed by Carpenter (1966), a harmonic orbit in the numerical sense, with respect to the rotating coordinate frame and about the L_4 triangular point of the earth-moon system is found. In addition, a nearly periodic orbit having half the period of the harmonic orbit is obtained.

2. THE MODEL AND COORDINATE SYSTEM

Consider an infinitesimal body of mass, m in a system of three bodies m_1, m_2, and m_3 which are the sun, earth, and moon respectively. Further assume that all four bodies remain in a plane so arranged that the center of mass, B of m_2 and m_3 is revolving around the center of mass, 0, of the entire system in a circular orbit and m_2 and m_3 themselves are revolving around B also in circular orbits. Table I indicates the numerical values used for this model, and Figure 1 shows the geometry.

TABLE I

Physical Characteristics of the Model and System

ITEM	REMARKS	SUN	MOON	REFERENCE ORBIT
Semimajor axis	Earth radii	23454.87	60.0	59.75609983
Eccentricity	Dimensionless	0.0	0.0	0.0
Argument of perigee	Degrees	0.0	0.0	0.0
Inclination	Degrees, with respect to X, Y plane	0.0	0.0	0.0
Longitude of the ascending node	Degrees	0.0	0.0	0.0
Mean anomaly at epoch	Degrees	0.0	0.0	60.0
Mass ratio	Body with respect to earth	332951.29	0.012294830	—
Mean motion	Radians per hour	$7.16754988 \times 10^{-4}$	$9.659527869 \times 10^{-3}$	$9.659527869 \times 10^{-3}$

Time is zero and epoch is defined as the instant the center of mass of the sun crosses the positive X axis.

Synodic period of the moon is taken to be 702.5992263 hours.

Earth's gravitational parameter, k_e^2, is taken to be 19.9094165 (earth radii)^3/hr^2.

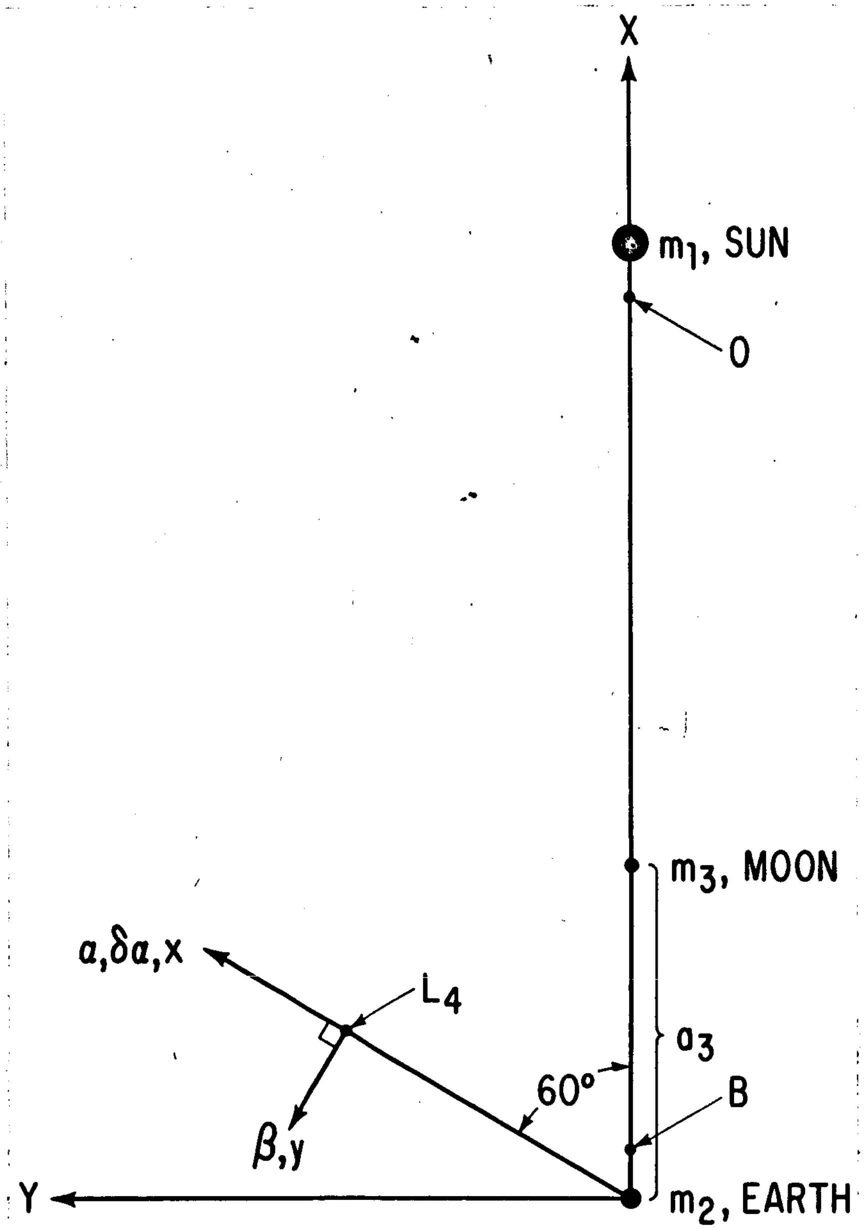

Figure 1. Geometry of the Coordinate System at Epoch

The right angle X, Y axis system with its origin at the center of mass of the earth is rotating at a uniform rate so as to keep the center of mass of the moon on the X axis. The masses m, m_1, m_2, and m_3 are in the X, Y plane. A point L_4, in the X, Y plane, is 60° from the X axis at lunar distance, a_3, and in advance of the moon's position. This point corresponds to a triangular point of the earth-moon system three body problem.

3. THE HARMONIC ORBIT

Using the model described the motion of an infinitesimal mass, m in the vicinity of the L_4 point was investigated with the purpose of obtaining a harmonic orbit in the numerical sense. This harmonic orbit has a period equal to the period of the disturbing force, which for this model is the moon's synodic period. Musen's (1963) method is applied with the perturbations represented in Chebyshev series as proposed by Carpenter (1966). In this method the geocentric position vector, $\vec{r}$, of the mass m near the L_4 triangular point is given by

$$\vec{r} = (1+\alpha)\vec{r}_0 + \beta\vec{w}$$

where α and β are the components of the perturbations, $\vec{r}_0$ is the position vector in the fixed reference ellipse, a is the semimajor axis of the reference ellipse and

$$\vec{w} = \frac{1}{n}\frac{d\vec{r}_0}{dt}$$

n being the mean motion in the reference ellipse and t the time. The functions α and β can be represented by uniformly convergent series in the interval $-1 \leq \tau \leq 1$ by

$$\alpha(\tau) = {\sum_{k=0}^{\infty}}' \alpha_k T_k(\tau)$$

$$\beta(\tau) = {\sum_{k=0}^{\infty}}' \beta_k T_k(\tau)$$

where the prime on the summation sign is used to indicate that the first term is to be factored by one-half. The $T_k(\tau)$ are the Chebyshev polynomials defined by

$$T_k(\tau) = \cos\left[k\cos^{-1}\tau\right]$$

where the coefficients of these polynomials are given by α_k and β_k.

The synodic period for the model being utilized is given by the equation

$$P = \frac{2\pi}{n_3 - n_1}$$

where n_1 and n_3 are the mean motions of the sun and moon respectively. The dimensionless time τ is related to time t from epoch by

$$\tau = \frac{2t}{P}$$

where t is zero at the epoch which is defined as the first time the mass m_1 crosses the positive X axis.

Starting with initial conditions at $\tau = 0$ in the X, Y plane and near the L_4 point, an orbit was generated by using the Chebyshev polynomial procedure. Initial conditions corresponding to τ of -1 and 1 were thus obtained. Using numerical partial derivatives from this orbit an iteration scheme was employed to match the initial conditions at τ of -1 and 1. After several iterations this was accomplished, and the results are shown in Table II. The α' and β' values, shown in this table, are derivatives of α and β with respect to nt. Relative geocentric errors in position and velocity are indicated by the differences in Table II. These differences indicate agreement in ten significant figures which correspond to changes from $\tau = -1$ to $\tau = 1$ of 0.2915 meters in the position vector,

TABLE II
Initial Conditions at $\tau = -1,\ 0,\ 1$ for the Harmonic Orbit

τ	$\alpha(\tau) \times 10^3$	$\beta(\tau) \times 10^3$	$\alpha'(\tau) \times 10^3$	$\beta'(\tau) \times 10^2$
-1	8.90721044	1.43123468	7.84005032	-1.341882564
0	4.41178027	16.95842648	14.7369698	-0.507181064
1	8.90721028	1.43123517	7.84005038	-1.341882534
Differences				
$\alpha(+1) - \alpha(-1) = -1.6 \times 10^{-10}$			$\alpha'(+1) - \alpha'(-1) = 6.0 \times 10^{-11}$	
$\beta(+1) - \beta(-1) = 4.9 \times 10^{-10}$			$\beta'(+1) - \beta'(-1) = 3.0 \times 10^{-10}$	

$\vec{r}$, and 1.12×10^{-6} meters per second in the velocity vector, $\dot{\vec{r}}$. From the numer-
point of view it seems adequate to call this a harmonic orbit.

Chebyshev coefficients for this orbit are given in Table III. Using the initial conditions at epoch this orbit was extended for a total of twelve synodic periods $(-1 \leq \tau \leq 25)$. This was done as a further check to insure the accuracy of the orbit. For this time period agreement with the initial orbit was eight significant figures in position and velocity. This is within the anticipated accuracy of the calculation. It was decided to plot the harmonic orbit with respect to a rotating coordinate system centered at L_4. Referring to Figure 2, a geocentric vector, $\vec{r}_0$, directed toward L_4 has a magnitude, a, defined by

$$a^3 = \frac{k_e^2 m_2}{n^2}$$

TABLE III
Chebyshev Coefficients for the Harmonic Orbit

k	$\alpha_k \times 10^6$	$\beta_k \times 10^6$
0	9233.8688	271.0928
1	-466.7498	4715.9669
2	4193.3102	-2590.7864
3	-2839.4208	-2764.2264
4	1530.9949	7775.0747
5	4927.4893	-3235.6624
6	-1877.9548	-5000.6993
7	-1940.7497	1552.0981
8	507.1676	1271.4117
9	356.8288	-301.5204
10	-67.8275	-172.1400
11	-41.2120	36.8027
12	4.4834	13.0026
13	4.3726	-3.8665
14	0.1851	-0.0154
15	-0.6623	0.4549
16	-0.0905	-0.1980
17	0.1213	-0.0468
18	0.0042	0.0448
19	-0.0196	-0.0019
20	0.0048	-0.0069
21	0.0024	0.0029
22	-0.0021	0.0007
23	-0.0001	-0.0009
24	0.0005	0.0000
25	-0.0001	0.0002
26	-0.0001	0.0000
27	0.0000	0.0000

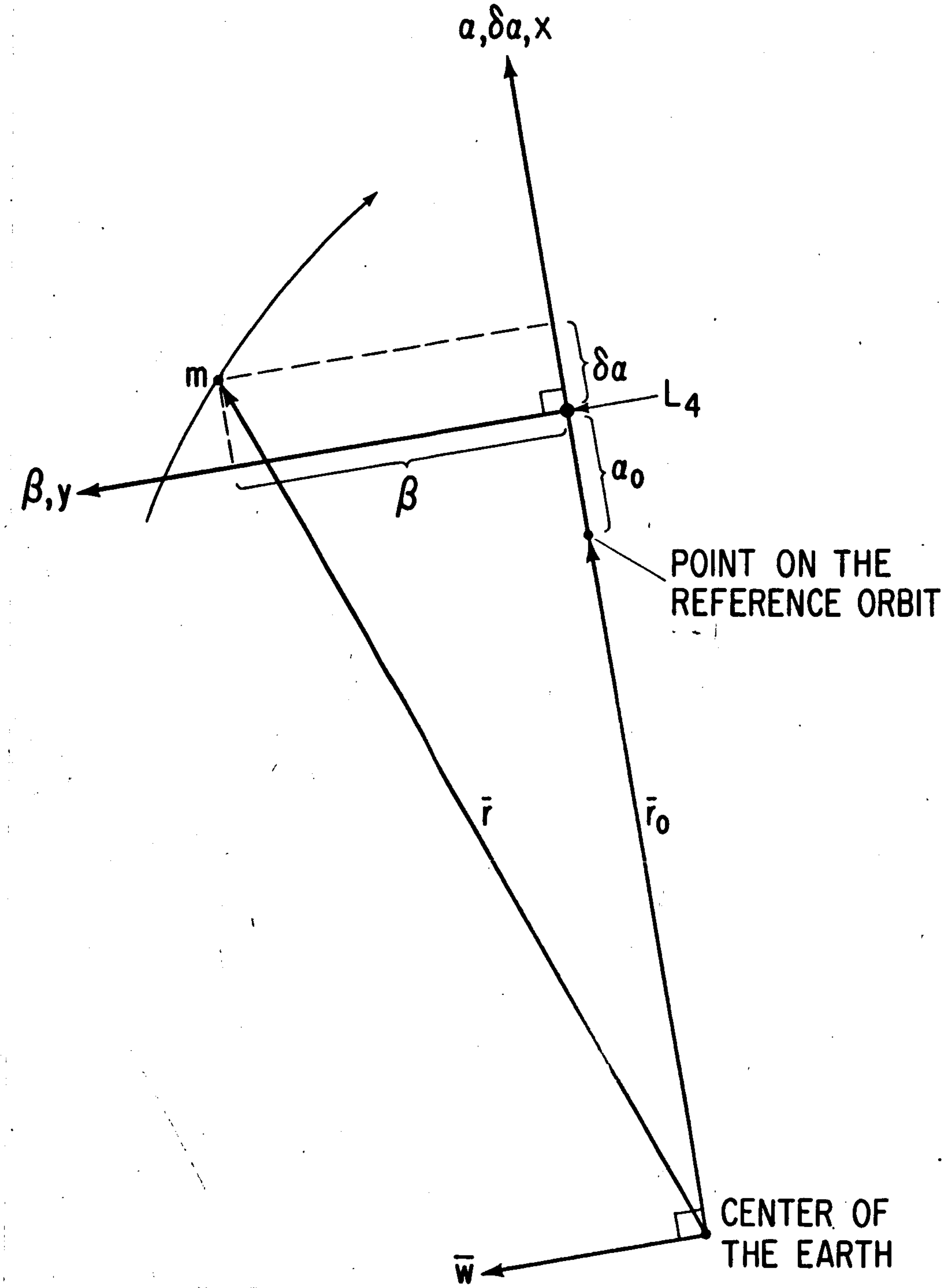

Figure 2. Coordinate Systems near L_4

where a is the semimajor axis of the reference orbit, k_e^2 is the earth's gravitational parameter and n is equal to the mean motion of the moon, n_3. The α, β coordinate system has its origin at $\vec{r}_0$ with α directed along $\vec{r}_0$ and β at right angles to α in the direction of motion in the reference orbit.

A $\delta\alpha$, β coordinate system has its origin at $(1 + \alpha_0)\,\vec{r}_0$ which corresponds to L_4. The value of α_0 is given by the quantity $(a_3 - a)/a$ where a_3, the moon's semi-major axis, is taken to be 60 earth radii.

The $\delta\alpha$ component is directed along $\vec{r}_0$ and β is the same component previously defined but translated parallel to itself to the L_4 point. Utilizing the data given in Table III the harmonic L_4 orbit is plotted, see Figure 3, using the $\delta\alpha$,

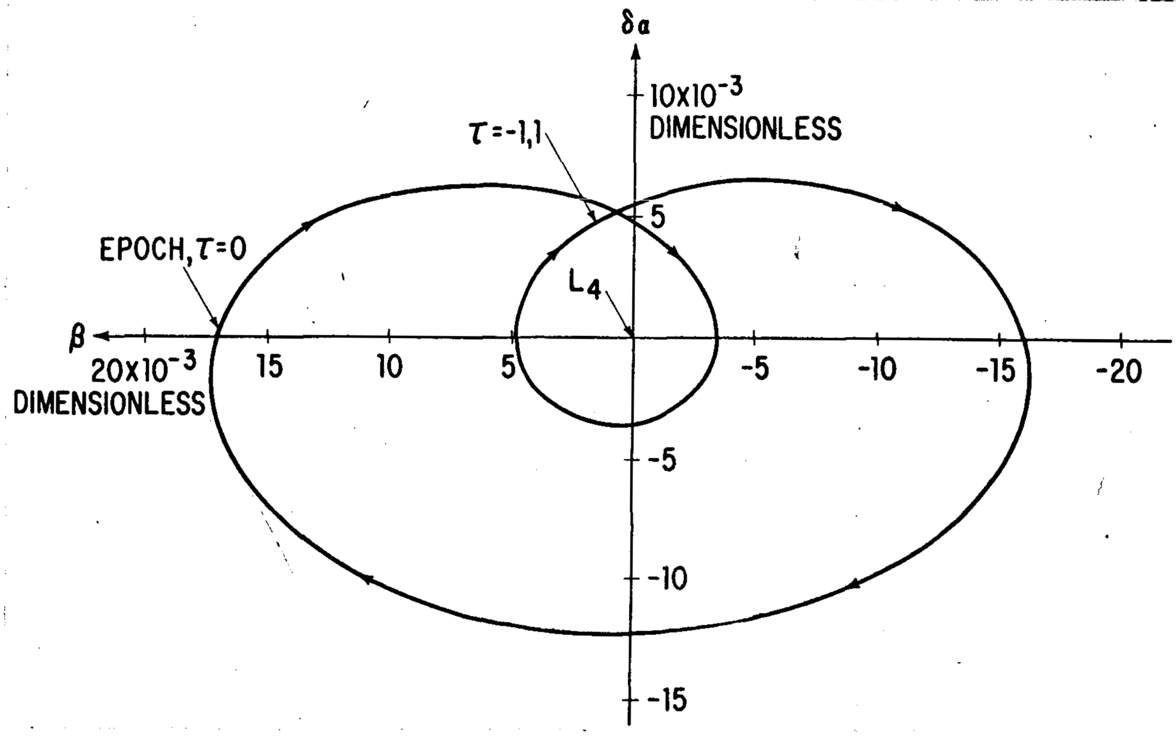

Figure 3. A Harmonic L_4 Orbit

β coordinate axes. Units for this plot are dimensionless with respect to the semimajor axis of the reference orbit. The harmonic orbit remains close to the L_4 point having a deviation of less than 1.8 percent (6860 km) of the earth-moon distance. This orbit is seen to describe two different size loops about L_4, the time in the smaller loop is 36 percent of the period. Conversion to the usual x, y coordinate system where x and y are centered at L_4 parallel to $\delta\alpha$, β but have dimensionless units with respect to the semimajor axis of the moon is accomplished by the transformation

$$x = \delta\alpha \frac{a}{a_3}$$

and

$$y = \beta \frac{a}{a_3}$$

Since the ratio (a_3/a) is near unity there would be no noticeable difference by replacing $\delta\alpha$ and β by x and y respectively in Figure 3, this matter being brought to attention for purposes of calculation.

4. A NEARLY PERIODIC ORBIT

Since the disturbing force is nearly periodic with half the synodic period of the moon, it is possible to find orbits which are nearly periodic with half the moon's synodic period. One such orbit was obtained by approximately matching initial conditions at τ of -1 and zero. The initial conditions are shown in Table IV. If the orbit were periodic with half the moon's synodic period, initial conditions at τ of -1 and zero would match with those at $\tau = 1$. The agreement in

TABLE IV
Initial Conditions at $\tau = -1, 0, 1$ for the Nearly Periodic Orbit

τ	$\alpha(\tau) \times 10^3$	$\beta(\tau) \times 10^3$	$\alpha'(\tau) \times 10^2$	$\beta'(\tau) \times 10^3$
-1	6.67701926	9.31381378	1.132682180	-9.25579538
0	6.67701357	9.31377754	1.132682017	-9.25577168
1	6.65506332	9.24996973	1.128066502	-9.18511281
Differences				
$\alpha(0) - \alpha(-1) = -5.69 \times 10^{-9}$			$\alpha'(0) - \alpha'(-1) = -1.63 \times 10^{-9}$	
$\beta(0) - \beta(-1) = -3.62 \times 10^{-8}$			$\beta'(0) - \beta'(-1) = 2.37 \times 10^{-8}$	
$\alpha(0) - \alpha(+1) = 2.19 \times 10^{-5}$			$\alpha'(0) - \alpha'(+1) = 4.615 \times 10^{-5}$	
$\beta(0) - \beta(+1) = 6.38 \times 10^{-5}$			$\beta'(0) - \beta'(+1) = -7.06 \times 10^{-5}$	

initial conditions between τ of -1 and zero is eight significant figures in position and velocity; however, between τ of zero and one there is only five significant figure agreement. Further reduction of this difference between τ of -1 and zero tended to increase the differences between zero and one. A plot of this nearly periodic L_4 orbit in the $\delta\alpha$, β coordinate system for τ between minus one and one is given in Figure 4. Also shown is the envelope of the orbit for six synodic periods ($-6 \leq \tau \leq 6$). During this time interval the orbit remains within 1.2 percent (4574 km) of the earth-moon distance from L_4. Deviations from the τ of minus one to one orbit are seen to be less than one tenth of a percent (381 km) of the earth-moon distance. During longer time intervals the deviations are expected to increase.

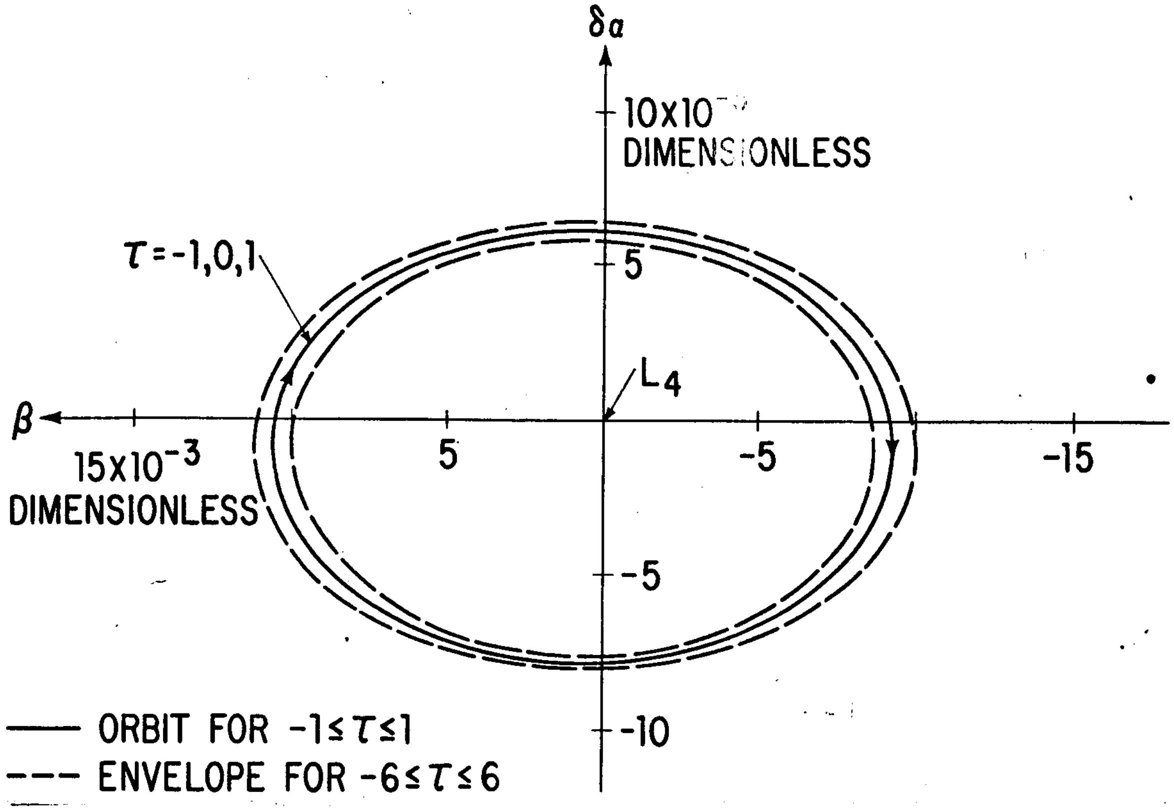

Figure 4. A Nearly Periodic L_4 Orbit

Using the Chebyshev polynomial approach, motion in the vicinity of L_4 can be investigated for other models of the earth-moon-sun system. One of the more interesting models would include the moon moving in an eccentric orbit. With this model some insight may be gained as to the importance of the moon's eccentricity on motion near L_4. The method is not restricted to simple models, e.g., it is possible to study motion near L_4 using the actual motions of the principal bodies taken from a general theory or from an ephemeris.

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